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BIOGRAPHY.

ARTEMAS MARTIN, M. A., PH. D., LL. D.

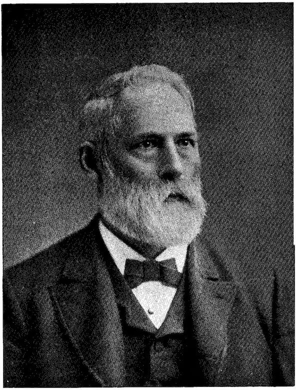
BY B. F. FINKEL.

The subject of this sketch was born in Steuben county, N. Y., August 3, 1835. Early, his parents moved to Venango county, Pa., where they lived for many years. Dr. Martin had no schooling in his early boyhood, except a little primary instruction; but by self-application and indefatigable energy which have told the story of many a great man, he has become familiar to every mathematician and lover of science in every civilized country of the world.

He was never a pupil at school, except when quite small, until in his fourteenth year. He had learned to read and write at home, but knew nothing of Arithmetic. At fourteen he commenced the study of Arithmetic, and after spending two winters in the district school, he commenced the study of Algebra. At seventeen, he studied Algebra, Geometry, Natural Philosophy, and Chemistry in the Franklin Select School, walking two and one-half miles night and morning. Three years after, he spent two and one-half months in the Franklin Academy, studying Algebra and Trigonometry. This finished his schooling. He taught district schools four winters, but not in succession. He was raised on a farm, and worked at farming and gardening in the summer; chopped wood in the winter; and after the discovery of oil in Venango county, worked at drilling oil wells a part of his time, always devoting his "spare moments" to study.

In the spring of 1869, the family moved to Erie county, Pa., where he resided until he entered the U. S. Coast Survey Office in 1885. While in Erie county, after 1871, he was engaged in market-gardening, which he carried on with great care and skill. He began his mathematical career when in his eighteenth year, by contributing solutions to the *Pittsburg Almanac*, soon after contributing problems to the "Riddler Column" of the Philadelphia *Saturday Evening Post*, and was one of the leading contributors for twenty years.

In the summer of 1864 he commenced contributing problems and solu-



ARTEMAS MARTIN, A. M., PH. D., LL. D.

tions to *Clark's School Visitor*, afterward the *Schoolday Magazine*, published in Philadelphia. In June, 1870, he took charge of the "Stairway Department" as editor, the mathematical department of which he had conducted for some years before. He continued in charge as mathematical editor till the magazine was sold to Scribner & Co., in the spring of 1875, at which time it was merged into "*St. Nicholas.*"

In September, 1875, he was chosen editor of a department of higher mathematics in the *Normal Monthly*, published by Professor Edward Brooks, Millersville, Pennsylvania, and held that position till the *Monthly* was discontinued in August, 1876. He published in the *Normal Monthly*, a series of sixteen articles on the Diophantine Analysis.

In June, 1877, Yale College conferred on him the honorary degree of Master of Arts (M. A.). In April, 1878, he was elected member of the London Mathematical Society. In June, 1882, Rutgers College conferred on him the honorary degree of Doctor of Philosophy (Ph.D.). March 7, 1884, he was elected a member of the Mathematical Society of France. In April, 1885, he was elected a member of the Edinburgh Mathematical Society. June 10, 1885, Hillsdale College conferred on him the honorary degree of Doctor of Laws (LL. D.). February 27, 1886, he was elected a member of the Philosophical Society of Washington. In June, 1881, he was elected Professor of Mathematics of the Normal School at Warrensburg, Missouri, but did not accept the position. November 14, 1885, Dr. Martin was appointed Librarian in the office of the U. S. Coast and Geodetic Survey. On August 27, 1889, he was elected a member, and on August 26, 1890, he was elected a Fellow of the American Association for the Advancement of Science. On April 3, 1891, he was elected a member of the New York Mathematical Society.

All these honors have been worthily bestowed and the Colleges and Societies conferring them have done honor to themselves in recognizing the merits of one who has become such a power in the scientific world through his own efforts.

He has contributed fine problems and solutions to the following journals of the United States: *School Visitor*, *Analyst*, *Annals of Mathematics*, *Mathematical Monthly*, *Illinois Teacher*, *Iowa Instructor*, *National Educator*, *Yates County Chronicle*, *Barnes' Educational Monthly*, *Wittenberger*, *Maine Farmers' Almanac*, *Mathematical Messenger*, *and Educational Notes and Queries*. Besides other contributions, he contributed thirteen articles on "Average" to the Mathematical Department of the *Wittenberger*, edited by Professor William Hoover. These are believed to be the first articles published on that subject in America.

Dr. Martin has also contributed to the following English mathematical periodicals: *Lady's* and *Gentleman's Diary*, *Messenger of Mathematics*, *Quarterly Journal of Mathematics*, and *The Educational Times* and *Reprint*.

The *Reprint* contains a large number of his solutions to difficult "Average" and "Probability" problems, which are master-pieces of mathemati-

cal thought and skill, and they will be lasting monuments to his memory. His style is direct, clear, and elegant. His solutions are neat, accurate, and simple. He has that rare and happy faculty of presenting his solutions in the simplest mathematical language, so that those who have mastered the elements of the various branches of mathematics, are able to understand his reasoning.

Dr. Martin is now (1894) editor of the *Mathematical Magazine*, and the *Mathematical Visitor*, two of the best mathematical periodicals published in America. These are handsomely arranged and profusely illustrated with very beautiful diagrams to the solutions, he doing the typesetting with his own hand. The typographical work of these journals is said to be the finest in America. The best mathematicians from all over the world contribute to these two journals. The *Mathematical Visitor* is devoted to Higher Mathematics, while the *Mathematical Magazine* is devoted to the solutions of problems of a more elementary nature. All solutions sent to Dr. Martin receive due credit, and, if it is possible to find room for them, the solutions are all published. He has thus encouraged many young aspirants to higher fields of mental activity. He is always ready to aid any one who is laboring to bring success with his work. He is of a kind and noble disposition and his generous nature is in full sympathy with every diligent student who is rising to planes of honor and distinction by self application and against adverse circumstances.

Dr. Martin has a large and valuable mathematical library containing many rare and interesting works. His collection of American arithmetics and algebras is one of the largest private collections of the kind in this country. He has also a large collection of foreign arithmetics and algebras; a large collection of grammars of the English language, mostly American; and a large collection of miscellaneous works, including botany, natural history, biography, poetry, American dictionaries, &c., and many curious books.

LOGARITHMS OF NEGATIVE NUMBERS.

By M. C. STEVENS, Professor of Higher Mathematics, Perdue University, Lafayette, Indiana.

On page 28, of B. O. Peirce's "Short Table of Integrals" is the following:—Log $(-u)=\log u+$ a constant. From the ordinary treatises on logarithms we learn that all possible numbers from $-\infty$ to $+\infty$ are required to express the logarithms of numbers from 0 to $+\infty$. The thoughtful student will therefore enquire, what is the nature of the constant which is added to the ordinary logarithm of u to give the log $(-u)$. The following explanation, although it is not new, may be interesting to some of the readers of the MONTHLY. I therefore submit it.

It is shown in many treatises on Higher Algebra, (See Wentworth's

College Alg. p. 492)that $e^{xi}=\cos x+i\ \sin x$, (i being used for the square root of -1). Since the trigonometric functions of $x+2a\pi$ are the same as for x (a being any integer) it follows that $e^{(x+2a\pi)i}=\cos(x+2a\pi)+i\sin(x+2a\pi)$.... (1). Making $x=\pi$, we have $e^{\pi(1+2a)i}=-1$. $\therefore \log(-1)=\pi(1+2a)i$.....(2).

If in equation (1) we make $x=0$ there results $e^{(2a\pi)i}=1$.

$\therefore$ $\log 1=(2a\pi)i$.....(3). If now we put $u=e^x$, then the ordinary $\log u=x$. But $-u=e^x.(-1)=e^{x+\pi(1+2a)i}$.

Whence, the general $\log(-u)=x+\pi(1+2a)i=$ *ordinary* $\log u+\pi(1+2a)i=\log u+\log(-1)$.

Thus it is shown that the constant in question is the impossible expression, $\pi(1+2a)i$, which by equation (2) is $\log(-1)$.

Again $u=e^x=e^x.(1)=e^{x+(2a\pi)i}$.

$\therefore$ It may be shown that the *general* $\log u=$ *ordinary* $\log u+(2a\pi)i$.

Hence, as a can have any integral value, it appears that any number, either positive or negative, can have an infinite number of logarithms; but only positive numbers can have their logarithms expressed by arithmetical numbers, and then only when $a=0$.

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M., (Princeton); Ph.D., (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

CHAPTER FIRST.

[Continued from the March Number.]

EUCLIDEANS.

John Wallis(1616—1703) deduces Euclid's Parallel-Postulate from the assumption of two similar figures, for example, two unequal triangles having their angles respectively equal and their sides proportional.

But we know that there is no need of assuming so much. The Parallel-Postulate follows from the existence of two unequal triangles of equal angle-sum, without regard to the proportionality of the sides or the respective equality of the angles.

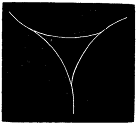
This remarkable truth was established in the very first Non-Euclidean Geometry published in 1733. After this date there existed in the world not only Euclid's, but also the two Non-Euclidean geometries; and so henceforth

the substitutes proposed for Euclid's postulatum lose interest, but I must still mention two, since they have come down to our very day.

One of these is a splendid illustration of the benefits of the study of non-Euclidean geometry, since though it deceived Sir Wm. Rowan Hamilton and John Casey, it would not now be accepted by the merest tyro who had looked into Lobatschewsky or Bolyai. It was not long ago exposed in *Nature* by that sound geometer O. Henrici, but I will quote its demolition by Perronet Thompson in 1833, who says of it: "Professor Playfair in the Notes to his 'Elements of Geometry' p. 409, has proposed another demonstration, founded on a remarkable *non causa procausa.*

It purports to collect that (on the sides being successively prolonged to the same hand) the exterior angles of a rectilineal triangle are together equal to four right angles, from the circumstance that a straight line carried round the perimeter of a triangle by being applied to all the sides in succession, is brought into its old situation again; the argument being, that because this line has made the sort of somerset it would do by being turned through four right angles about a fixed point, the exterior angles of the triangle have necessarily been equal to four right angles.

The answer to which is, that there is no connexion between the things at all, and that the result will just as much take place where the exterior angles are avowedly not equal to four right angles. Take, for example, the plane triangle formed by three small arcs of the same or equal circles, as in the margin; and it is manifest that an arc of this circle may be carried round precisely in the way described and return to its old situation, and yet there be no pretense for inferring that the exterior angles were equal to four right angles.



And if it is urged [as actually was by John Casey] that these are *curved* lines and the statement made was of *straight*; then the answer is by demanding to know what property of straight lines has been laid down or established, which determines that what is not true in the case of other lines is true in theirs.

It has been shown that, as a general proposition, the connextion between a line returning to its place and the exterior angles having been equal to four right angles, is a *non sequitur*; that it is a thing which may not be; that the notion that it returns to its place *because* the exterior angles have been equal to four right angles, is a mistake. From which it is a legitimate conclusion that if it has pleased nature to make the exterior angles of a triangle greater or less than four right angles, this would not have created the smallest impediment to the line's returning to its old situation after being carried round the sides; and consequently the line's returning is no evidence of the angles not being greater or less than four right angles."

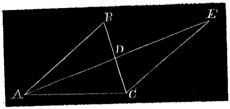
The other fallacy which I shall now proceed to mention occurred first in the Twelfth Edition of Legendre's geometry, but its viciousness having been

recognized by its author, it was afterwards withdrawn. It is, to say the least, queer, that this error, exploded and known as a fallacy for more than sixty years, should in 1892 have been reproduced by a paradoxer named J. N. Lyle, in a pamphlet printed by him in St. Louis.

Lobatschewsky's § 19 is (See Halsted's translation, 4th Ed. p. 16), *In a rectilineal triangle the sum of the three angles can not be greater than two right angles.*

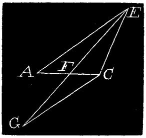
On the assumption that the straight is infinite, and that two straights which have crossed never recur or meet again, this is deduced by Lobatschewsky as follows:

Suppose in the triangle ABC the sum of the three angles is equal to $\pi+a$; then choose in case of the inequality of the sides the smallest BC, halve it in D, draw from A through D the line AD and make the prolongation of it, DE, equal to AD, then join the point E to the point C by the straight line EC.



In the congruent triangles ADB and CDE, the angle $ABD=DCE$, and $BAD=DEC$ (Theorems 6 and 10); whence follows that also in the triangle ACE the sum of the three angles must be equal to $\pi+a$.

But also the smallest angle BAC (Theorem 9) of the triangle ABC in passing over into the new triangle ACE has been cut up into the two parts EAC and AEC; and the larger of the two angles ABC, BAC, namely ABC, has been taken away from the sum of these two, ABC and BAC, and added to the angle ACB. Now again in the new triangle ACE, halve the smallest side AC in F, draw from E through F the line EF and make the prolongation of it, FG, equal to EF, then join the point G to the point C by the straight line GC.



Thus we take away from the sum of the angles EAC and AEC the greater EAC and add it to the angle ACE.

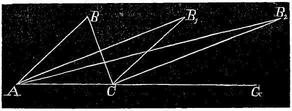
Continuing this process, continually halving the side opposite the smallest angle, we must finally attain to a triangle in which the sum of the three angles is $\pi+a$, but wherein are two angles, each of which in absolute magnitude is less than $\frac{1}{2}a$; since now however the third angle cannot be greater than π, so must a be either null or negative.

This demonstration by Lobatschewsky is in perfect accord with Euclid, since it is based upon the first proposition of Euclid's Tenth Book: *If from the greater of two unequal magnitudes there be taken more than its half, and from the remainder more than its half, and so on, there shall at length remain a magnitude less than the smaller of the proposed magnitudes.*

Here the smaller of the proposed magnitudes is $\frac{1}{2}a$, and the greater is the sum of ABC and BAC.

Now our paradoxer, Mr. Lyle, read this demonstration in my translation, and though, as we shall see, he did not understand it, yet he felt that because Lobatschewsky gave it, it must be rigorous, and so thought he might trust himself to reproduce it, but in doing so, laid bare in the most ludicrous manner, his utter failure to grasp it or to see its cogency.

He calls his blunder "Lobatschewsky's Theorem 19 recast in the rigidest (sic) Euclidian forms," and uses for his fallacious pseudo-proof this figure which at once tells us how he is about to lay bare his mental nakedness. Not understanding the proof, of course he does not comprehend that he must never halve the greater of two sides going to C, as AC, and CB_2; so as it happens he *always* halves the greater, thus repeating his blunder an indefinite number of times, and utterly vitiating a beautiful proof.



Instead of taking from the sum of ABC and BAC more than half and continually more than half of the remainder, he only takes a little piece that rapidly becomes less than a millionth part of their sum, while his seeming highway of demonstration dwindles to a squirrel-path and runs up a tree.

Nothing daunted, he reproduces on his next page the well-known fallacious demonstration from the twelfth edition of Legendre's geometry, that the sum of the three angles of a rectilineal triangle cannot be less than two right angles; only he makes rediculous, by his "recast," what was always erroneous.

Lobatschewsky had this disgraceful fallacy of Legendre's before him when he wrote in 1840 the book I have translated, so I need only quote in regard to it a few sentences written in 1833 by Perronet Thompson. Referring to the above valid procedure as I have given it, he says: "the described process may be continued, till two of the angles of the last resulting triangle are together less than any magnitude that shall have been assigned; and consequently the third or remaining angle may be made to approach, within any magnitude however small it may be chosen to assign, to *the sum of the three angles of the original or any of the intervening triangles.* All this is irrefragable; but not so the proposition next taken for granted, which is that the third angle last mentioned approaches within any magnitude however small it may be chosen to assign, to *the sum of two right angles*".... "The conclusion is founded on neglect of the very early mathematical truth, that continually increasing is no evidence of ever arriving at a magnitude assigned."

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REMARKS ON PROFESSOR LYLE'S POSTULATE I. OF EUCLID'S ELEMENTS.

By JOHN DOLMAN, Jr., Counsellor at Law, Philadelphia, Pennsylvania.

Professor Lyle, in No. 1 of THE AMERICAN MATHEMATICAL MONTHLY, falls into an error, through misapprehending the meaning of Lobatschewsky.

According to Lobatschewsky the angle-sum of a rectilineal triangle decreases as the area of the triangle increases, but is always less than two right angles.

Lobatschewsky' sgeometry does not apply to the plane, nor to space as we know it, but to what has since been termed a pseudospherical surface,or one of uniform negative curvature in the same sense that the surface of a sphere is of uniform positive curvature. Such a surface cannot be fully constructed,and the theorems of Lobatschewsky are seemingly impossible; but his geometry is consistant with itself and contradicts none of the postulates or axioms of Euclid except the 12th. His straight line is not, (it is true,) really straight, but is the shortest distance between two points, and lying wholly in the given space. A straight line may be drawn between any two points in the space, and a triangle can be formed of three straight lines joining any three points.

This being premised, the Professor's first error is in defining a finite straight line as one that has two ends, and in confounding "infinite" and "boundless". He may refer to a *terminated* straight line, and his definition is then correct.

Now, it is true, a straight line can be drawn from any point in AC to any point in CB, and the triangle ECF will have an angle-sum greater than two right angles$-a$. This however, is not contrary to the hypothesis that the angle-sum shall be less than two right angles. No matter how small a is taken it can still be divided. Though the angle C be as nearly equal to two right angles as you choose yet E and F, taken together, will not entirely make up the difference. If a is taken infinitely small the area of the triangle ECF will be infinitely small, and its angle-sum will differ from two right angles by less than any assignable quantity.

MORE REMARKS ON DIVISION.

By J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pennsylvania.

An introduction of half a hundred lines from ancient history overshadowing a mere assertion as a conclusion, might give these remarks an air of profundity, but that species of pedantry, being an "idol" which neither Gauss nor Argand has yet knocked down, is a pet whose sacred form profane hands must not pollute.

Mr. Smith says that multiplication was, originally, a mere process of adding; but what is mathematically true today was true from the beginning and always will be true. It is thus seen that Mr. Smith either corroborates my statement as to multiplication, or denies the fixed unchangeable character of mathematical truth.

When children are learning multiplication of integers in arithmetic they *do* see, they *can* see, they are *asked* to see but one explanation, viz., that multiplication is a process of adding. The same language will not explain multiplication by a fraction.

In *general*, multiplication is finding one of the four terms of a proportion; *i. e.*, it is finding a number that bears the same relation to a multiplicand as the multiplier, or operator, bears to unity. The *arithmetical* conception of multiplication by integers admits but one definition of the process, which is that almost universally given in the text-books—a process of adding. The arithmetical conception of discrete number does not enable us to locate $\sqrt{3}$, although it is a real number between 1 and 2. To divide 10 into two parts whose products shall be 40 is impossible by arithmetic. That is to say, what is legitimate and possible in an algebraic or geometric sense may be impossible in an arithmetical sense.

In the domain of pure arithmetic $\sqrt{-15}$ is no more absurd than $\$12 \div 2$.

A *concrete number* in arithmetic is, in the child's mind, a number of objects, as 8 books. These may be divided *physically* into 2 equal parts, but 8 books÷2 as an arithmetical operation is absurd. We may divide x^2 by x, but not by y. The operation may be indicated $\left(\frac{x^2}{y}\right)$ but can not be *performed.* Ten 5's can not be divided by five 10's unless both be reduced to the same unit.

Multiplication is *one* thing only, and division, its inverse, can be but one thing. If division is a process of finding how often one quantity is contained in another, it cannot at the same time be a process of finding one of the equal parts of a quantity. The latter is an application of division.

In a recent issue of the *Popular Educator* Dr. McLellan, author of "Applied Psychology," etc., devotes several columns to proving that a concrete number can be divided by an abstract. His whole argument is based upon the commutative law of multiplication, which Mr. Smith has consigned to the "museum of antiquities"! Without the "old idol," the commutative law, it is not possible to prove that \$12 can be divided by 2, in an arithmetical sense.

While Mr. Smith would appear as an exponent of progress, he is championing an "old fogy" notion. Before he was born arithmetics taught that $\$12 \div 2 = \$6$, but recent authors have advanced a step, and, strange as it may seem, he dons his knightly armor to do battle with the progressive idea. Since the commutative law of multiplication has been duly labeled and placed in the "museum," does Mr. Smith intend to "do or die" in defence of its relative—the commutative law of division ?

We find that $\frac{1}{2}$ of $\$12 = \$6$. It will be noticed that the 12 has been divid-

ed, the \$ has *not*. The \$ has not entered into the division at all. It is simply annexed to the 6. The principle is evident.

ARE DIFFERENTIALS FINITE QUANTITIES?

By JOHN N. LYLE, Ph. D., Professor of Mathematics, Westminster College, Fulton, Missouri.

In seeking for a correct answer to the above question let us reconsider in detail a familiar elementary example.

Let $u=x^2 \dots (1)$, in which u is a function of the independent variable that increases in value *uniformly*.

The *increment* of the variable x in *a unit of time* is the *rate of variation* of the variable x and may be appropriately represented by the symbol dx.

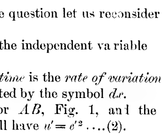
When the variable x reaches the value x' or AB, Fig. 1, and the function w, the corresponding value u' or AC we shall have $u'=x'^2 \dots (2)$.

Let $\Delta x=\frac{dx}{n}$, in which Δx represents the increment of the variable x in $\frac{1}{n}$ of a unit of time. Since x varies uniformly, $n\times\Delta x$ will equal dx; that is, will equal the rate of variation of the independent variable x.

When the variable x reaches the value x''; that is, $x'+\frac{dx}{n}$, or Aa, and the function u, the corresponding value u'' or Ai we shall have $u''=x''^2=\left(x'+\frac{dx}{n}\right)^2 \dots (3)$.

Subtract (2) from (3).

Then $u''-u'$, or $\Delta u'=x''^2-x'^2=2x'\frac{dx}{n}+\frac{dx^2}{n^2} \dots (4)$.



Multiply both members by n. Then $n\times\Delta u'=2x'dx+\frac{dx^2}{n} \dots (5)$. The second member of (5) is made up of two parts, one of which, $2x'dx$, is constant whatever the value of n may be, and the other, $\frac{dx^2}{n}$, decreases without limit as n increases without limit. $\frac{dx^2}{n}$ is n times $\frac{dx^2}{n^2}$, the addition to the increment of the function in $\frac{1}{n}$ of the unit of time due to the *increase* in the tendency of the function to vary after passing the value u'.

$\frac{2x'dx}{n}\times n$, that is, $2x'dx$ is the increment of the function u in the unit of time that is due to its tendency to vary when it reaches the value u'.

The increment $2x'dx$ is received *uniformly* during the unit of time

and symbolizes the rate of variation of the function u when it reaches the value u'.

Let $i=\frac{dx^2}{n}$....(6). Subtract (6) from (5).

Then $n\times\Delta u'-i=2x'dx$.....(7). Since $2x'dx$ symbolizes the rate of variation of the function u corresponding to the value u',

$n\times\Delta u'-i$ must symbolize the same thing.

Instead of $n\times\Delta u'-i$ write du' and we shall have $du'=2x'dx$...(8), in which du' represents the rate of variation of the function u corresponding to the value u'. du' is, also, seen to be the limit that the variable product $n\times\Delta u'$ approaches as n increases without limit and $\Delta u'$ decreases without limit.

The geometrical value of dx is the line BE or DG in Fig. 1. But each of these lines has two ends and is, therefore, *finite*. The geometrical value of x' is the line BC or DC.

As each of these lines has *two ends* they are *finite*.

It follows that $2x'dx$, that is, du' is *finite*. The geometrical expression for $2x'dx$ is the sum of the two rectangles CE and CG.

By dropping the primes in (8) we have the general expression $du=2xdx$....(9), in which dx stands for the differential or rate of variation of the function u.

By hypothesis dx is *a constant*. Furthermore, the values x', x'', x''', &c., attributed to the variable x are *finite*. Hence, the values of du corresponding to the different values of the variable x are *finite*.

Remark 1. $\Delta u'$ is the increment of the function u in $\frac{1}{n}$ of the unit of time after reaching the value u'.

The geometrical value of $\Delta u'$ is the sum of the two rectangles Ca and Ce and the square Ci.

$n\times Ca=CE$, $n\times Ce=CG$, and $n\times Ci=CO$.

Equation (5) is obtained by multiplying the increment of the function u in $\frac{1}{n}$ of the unit of time by n. This is the data from which we deduce the inference that the rate of variation of the function u corresponding to the value u' is $2x'dx$, the limit of $n\times\Delta u'$ as n increases without limit and du diminishes without limit.

Since it would be absurd to destroy the premises of an argument in order to reach a logical conclusion therefrom, so it would be illogical and absurd to annihilate $\Delta u'$ in order to obtain du'.

Remark 2. A *finite* straight line is one that has a beginning and a termination, that is, two ends.

A *finite* series is one that has a first and a last term.

A *finite* number is one whose units would constitute a series having a first and a last term.

Remark 3. Since the geometrical value of dx is the finite line BE, $\frac{1}{n}$

of dx, or Δx, is a part of BE having two ends and is, therefore, *finite*.

If the successive values of n are the numbers 1, 2, 3, 4, 5, &c., of the counting series, they are finite numbers inasmuch as the units of eacn of these numbers would constitute a series having a first and a last term.

Remark 4. Ci is the geometrical equivalent of $\frac{dx^2}{n^2}$, or Δx^2; and $\times Ci$ or CO is the geometrical equvivalent of $\frac{dx^2}{n}$, or $n\times\Delta x^2$.

As n increases without limit Ci and CO diminish without limit.

The rectangles Ci and CO have finite areas inasmuch as their sides are finite lines. Diminishing their value neither annihilates them nor changes their character as finite quantities.

Remark 5. Let $x=12$ inches, $dx=1$ inch and $n=10$.

Then will $\Delta u'=24\times.1+.1^2=2.41$, and $n\times\Delta u'=10\times2.41=24.1$.

If $n=100$: We have $\Delta u'=24\times.01+.01^2=.2401$, and $n\times\Delta u'=100\times.2401=24.01$.

If $n=1000$: Then $\Delta u'=24\times.001+.001^2=.024001$, and $n\times\Delta u'=1000\times.024001=24.001$.

The variable product $n\times\Delta u'$ whose successive values are 24.1, 24.01, 24.001, &c., approaches the limit 24 which is the rate of increase of the area when the side is 12 inches and its rate 1 inch per second. du' is the limit of $n\times\Delta u'$ and its value is 24.

The numerical values attributed to n and Δx are, likewise, *finite*.

Remark 6. Equation (7) is derived from equation (5) in accordance with Euclid's axiom, 3,—If equals be taken from equals, the remainders are equal.

Berkeley, Carnot, and Compte refer to a compensation of errors in the Calculus.

This compensation is really accounted for by Euclid's axiom 3, since *i* the difference between $n\times\Delta u'$ and du' is equal to $\frac{dx^2}{n}$ the difference between the second member of (5) and $2x'dx$.

NOTE ON THE CENTROID OF PLANE AREAS.

By Professor G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

Let it be required to find the centroid of the area of the curve $r^{\frac{1}{m}}=a^{\frac{1}{m}}\cos\frac{\theta}{m}$, when the density varies as the nth power of the distance of any point from the origin.

From works on Statics we get

$$\bar{x}=\frac{\int_{a}^{\beta}\int_{0}^{r} k\rho r_1^{2}\cos\theta dr_1 d\theta}{\int_{a}^{\beta}\int_{0}^{r} k\rho r_1 dr_1 d\theta}$$

where ρ is the density and k the thickness of the lamina at any element.

In this case $\rho=\mu r_1^{n}$, and k is constant. The curve is also symmetrical to the prime vector; therefore, the abscissa being the same for the whole curve as for the half above the axis,

$$\bar{x}=\left[\int_{0}^{\frac{m\pi}{2}}\int_{0}^{r} r_1^{n+2}\cos\theta dr_1 d\theta\right] \Big/ \left[\int_{0}^{\frac{m\pi}{2}}\int_{0}^{r} r_1^{n+1} dr_1 d\theta\right]$$

$$=\frac{n+2}{n+3}\left[\int_{0}^{\frac{1}{2}m\pi} r^{n+3}\cos\theta d\theta\right] \Big/ \left[\int_{0}^{\frac{1}{2}m\pi} r^{n+2}\, d\theta\right]$$

But $r=a\cos^{m}\frac{\theta}{m}$. Let $\frac{\theta}{m}=\varphi$, or $\theta=m\varphi$, then we get

$$\bar{x}=\frac{n+2}{n+3}a\left[\int_{0}^{\frac{1}{2}\pi}\cos^{mn+3m}\varphi\cos m\varphi d\varphi\right] \Big/ \left[\int_{0}^{\frac{1}{2}\pi}\cos^{mn+2m}\varphi d\varphi\right]$$

Now when m is an odd integer, and positive, $\cos m\varphi=A_1\cos\varphi+A_2\cos^3\varphi + A_3\cos^5\varphi+A_4\cos^7\varphi+A_5\cos^9\varphi+A_6\cos^{11}\varphi+A_7\cos^{13}\varphi+A_8\cos^{15}\varphi + A_9\cos^{17}\varphi+A_{10}\cos^{19}\varphi+\ldots\ldots,$

where $A_1=\frac{m}{(-1)^{\frac{m-1}{2}}}$, $A_2=-\frac{m(m^2-1)}{(-1)^{\frac{m-1}{2}}\underline{|3}}$, $A_3=\frac{m(m^2-1)(m^2-3^2)}{(-1)^{\frac{m-1}{2}}\underline{|5}}$, &c.

$$\therefore\ \bar{x}=\frac{n+2}{n+3}a\left[\int_{0}^{\frac{1}{2}\pi}\left(A_1\cos^{mn+3m+1}\varphi+A_2\cos^{mn+3m+3}\varphi+A_3\cos^{mn+3m+5}\varphi\right.\right.$$

$$+A_4\cos^{mn+3m+7}\varphi+A_5\cos^{mn+3m+9}\varphi+A_6\cos^{mn+3m+11}\varphi$$

$$+A_7\cos^{mn+3m+13}\varphi+A_8\cos^{mn+3m+15}\varphi+A_9\cos^{mn+3m+17}\varphi$$

$$\left.\left.+A_{10}\cos^{mn+3m+19}\varphi+\ldots\right)d\varphi\right]\Big/\left[\int_{0}^{\frac{1}{2}\pi}\cos^{mn+2m}\varphi d\varphi\right].$$

$$\bar{x}=\frac{n+2}{n+3}a\left[A_1\frac{\Gamma\left(\frac{mn+3m+2}{2}\right)}{\Gamma\left(\frac{mn+3m+3}{2}\right)}+A_2\frac{\Gamma\left(\frac{mn+3m+4}{2}\right)}{\Gamma\left(\frac{mn+3m+5}{2}\right)}+A_3\frac{\Gamma\left(\frac{mn+3m+6}{2}\right)}{\Gamma\left(\frac{mn+3m+7}{2}\right)}\right.$$

$$+A_4\frac{\Gamma\left(\frac{mn+3m+8}{2}\right)}{\Gamma\left(\frac{mn+3m+9}{2}\right)}+A_5\frac{\Gamma\left(\frac{mn+3m+10}{2}\right)}{\Gamma\left(\frac{mn+3m+11}{2}\right)}+A_6\frac{\Gamma\left(\frac{mn+3m+12}{2}\right)}{\Gamma\left(\frac{mn+3m+13}{2}\right)}$$

$$A_7\frac{\Gamma\left(\frac{mn+3m+14}{2}\right)}{\Gamma\left(\frac{mn+3m+15}{2}\right)}+A_8\frac{\Gamma\left(\frac{mn+3m+16}{2}\right)}{\Gamma\left(\frac{mn+2m+17}{2}\right)}+A_9\frac{\Gamma\left(\frac{mn+3m+18}{2}\right)}{\Gamma\left(\frac{mn+3m+19}{2}\right)}$$

$$+A_{10}\frac{\Gamma\left(\frac{mn+3m+20}{2}\right)}{\Gamma\left(\frac{mn+3m+21}{2}\right)}+\ldots\Bigg] \Bigg/ \left[\frac{\Gamma\left(\frac{mn+2m+1}{2}\right)}{\Gamma\left(\frac{mn+2m+2}{2}\right)}\right]$$

Let $m=1$, then

$$\bar{x}=\frac{n+2}{n+3}a\left[\frac{\Gamma\left(\frac{n+5}{2}\right)}{\Gamma\left(\frac{n+6}{2}\right)}\right]\Bigg/\left[\frac{\Gamma\left(\frac{n+3}{2}\right)}{\Gamma\left(\frac{n+4}{2}\right)}\right]=\frac{n+2}{n+3}\left[\frac{n+3}{n+4}\right]a=\frac{n+2}{n+4}a.$$

When $n=0$, $\bar{x}=\frac{1}{2}a$ $\therefore$ the centre of the circle is its centroid.

Let $m=3$, then

$$\bar{x}=\frac{n+2}{n+3}a\left[4\frac{\Gamma\left(\frac{3n+13}{2}\right)}{\Gamma\left(\frac{3n+14}{2}\right)}-3\frac{\Gamma\left(\frac{3n+11}{2}\right)}{\Gamma\left(\frac{3n+12}{2}\right)}\right]\Bigg/\left[\frac{\Gamma\left(\frac{3n+7}{2}\right)}{\Gamma\left(\frac{3n+8}{2}\right)}\right]$$

$$\bar{x}=\frac{n+2}{n+3}\left[4\frac{(3n+11)(3n+9)(3n+7)}{(3n+12)(3n+10)(3n+8)}-3\frac{(3n+9)(3n+7)}{(3n+10)(3n+8)}\right]a$$

$$=\frac{n+2}{n+3}\left[\frac{(3n+9)\ (3n+7)}{(3n+12)(3n+10)}\right]a=\frac{(n+2)(3n+7)}{(n+4)(3n+10)}a$$

Let $m=5$, then

$$\bar{x}=\frac{n+2}{n+3}a\left[5\frac{\Gamma\left(\frac{5n+17}{2}\right)}{\Gamma\left(\frac{5n+18}{2}\right)}-20\frac{\Gamma\left(\frac{5n+19}{2}\right)}{\Gamma\left(\frac{5n+20}{2}\right)}+16\frac{\Gamma\left(\frac{5n+21}{2}\right)}{\Gamma\left(\frac{5n+22}{2}\right)}\right]\Bigg/\left[\frac{\Gamma\left(\frac{5n+11}{2}\right)}{\Gamma\left(\frac{5n+12}{2}\right)}\right]$$

$$\bar{x}=\frac{n+2}{n+3}\left[5\frac{(5n+15)(5n+13)(5n+11)}{(5n+16)(5n+14)(5n+12)}-20\,\frac{(5n+17)(5n+15)(5n+13)(5n+11)}{(5n+18)(5n+16)(5n+14)(5n+12)}\right.$$

$$\left.+16\,\frac{(5n+19)(5n+17)(5n+15)(5n+13)(5n+11)}{(5n+20)(5n+18)(5n+16)(5n+14)(5n+12)}\right]a$$

$$=\bar{x}\frac{n+2}{n+3}\left[\frac{(5n+15)(5n+13)(5n+11)}{(5n+20)(5n+18)(5n+16)}\right]a=\frac{(n+2)(5n+11)(5n+13)}{(n+4)(5n+16)(5n+18)}a$$

The rule is now established. The first factor is always $\frac{n+2}{n+4}$, the numerator of the second factor is $mn+2m+1$ the denominator is $mn+3m+1$, the numerator of each succeeding factor is 2 greater than the one immediately preceding, so also for each succeeding factor in the denominator.

Let $m=7$, then $mn+2m+1=7n+15$, and $mn+3m+1=7n+22$ and

$$\bar{x}=\frac{(n+2)(7n+15)(7n+17)(7n+19)}{(n+4)(7n+22)(7n+24)(7n+26)}a$$

The number of factors when $m=1$, is 1, for $m=3$, is 2, for $m=5$, is 3 for $m=7$, is 4 and so on,

Let $m=9$, then $mn+2m+1=9n+19$; $mn+3m+1=9n+28$.

$$\therefore\ \bar{x}=\frac{(n+2)(9n+19)(9n+21)(9n+23)(9n+25)}{(n+4)(9n+28)(9n+30)(9n+32)(9n+34)}a,$$ and so on for any value of m.

The centroid is evidently on the axis of symmetry. $\therefore\ \bar{y}=0$.

When $m=9$, $\bar{x}$ reduces to the following

$$\bar{x}=\frac{(n+2)(3n+7)(9n+19)(9n+23)(9n+25)}{(n+4)(3n+10)(9n+28)(9n+32)(9n+34)}a.$$

When m is an even integer, and positive,

$$\cos m\varphi=A_1+A_2\cos^2\varphi+A_3\cos^4\varphi+A_4\cos^6\varphi+A_5\cos^8\varphi+A_6\cos^{10}\varphi$$
$$+A_7\cos^{12}\varphi+A_8\cos^{14}\varphi+A_9\cos^{16}\varphi+A_{10}\cos^{18}\varphi+\dots,$$

where $A_1=\frac{1}{(-1)^{\frac{m}{2}}}$, $A_2=-\frac{m^2}{(-1)^{\frac{m}{2}}2}$ $A_3=\frac{m^2(m^2-2^2)}{(-1)^{\frac{m}{2}}\lfloor 4}$ &c.

$$\bar{x}=\frac{n+2}{n+3}a\Big[\int_0^{\frac{1}{2}\pi}\Big(A_1\cos^{mn+3m}\varphi+A_2\cos^{mn+3m+2}\varphi+A_3\cos^{mn+3m+4}\varphi$$
$$+A_4\cos^{mn+3m+6}\varphi+A_5\cos^{mn+3m+8}\varphi+A_6\cos^{mn+3m+10}\varphi+A_7\cos^{mn+3m+12}\varphi$$
$$+A_8\cos^{mn+3m+14}\varphi+A_9\cos^{mn+3m+16}\varphi+$$
$$A_{10}\cos^{mn+3m+18}\varphi+\dots\Big)d\varphi\Big]\Big/\Big[\int_0^{\frac{1}{2}\pi}\cos^{mn+2m}\varphi d\varphi\Big].$$

$$\bar{x}=\frac{n+2}{n+3}a\Big[A_1\frac{\Gamma\left(\frac{mn+3m+1}{2}\right)}{\Gamma\left(\frac{mn+3m+2}{2}\right)}+A_2\frac{\Gamma\left(\frac{mn+3m+3}{2}\right)}{\Gamma\left(\frac{mn+3m+4}{2}\right)}+A_3\frac{\Gamma\left(\frac{mn+3m+5}{2}\right)}{\Gamma\left(\frac{mn+3m+6}{2}\right)}$$

$$+A_4\frac{\Gamma\left(\frac{mn+3m+7}{2}\right)}{\Gamma\left(\frac{mn+3m+8}{2}\right)}+A_5\frac{\Gamma\left(\frac{mn+3m+9}{2}\right)}{\Gamma\left(\frac{mn+3m+10}{2}\right)}+A_6\frac{\Gamma\left(\frac{mn+3m+11}{2}\right)}{\Gamma\left(\frac{mn+3m+12}{2}\right)}$$

$$+A_7\frac{\Gamma\left(\frac{mn+3m+13}{2}\right)}{\Gamma\left(\frac{mn+3m+14}{2}\right)}+A_8\frac{\Gamma\left(\frac{mn+3m+15}{2}\right)}{\Gamma\left(\frac{mn+2m+16}{2}\right)}+A_9\frac{\Gamma\left(\frac{mn+3m+17}{2}\right)}{\Gamma\left(\frac{mn+3m+18}{2}\right)}$$

$$+A_{10}\frac{\Gamma\left(\frac{mn+3m+19}{2}\right)}{\Gamma\left(\frac{mn+3m+20}{2}\right)}+\dots\Bigg]\Bigg/\left[\frac{\Gamma\left(\frac{mn+2m+1}{2}\right)}{\Gamma\left(\frac{mn+2m+2}{2}\right)}\right].$$

Let $m=2$, then

$$x=\frac{n+2}{n+3}a\left[2\frac{\Gamma\left(\frac{2n+9}{2}\right)}{\Gamma\left(\frac{2n+10}{2}\right)}-\frac{\Gamma\left(\frac{2n+7}{2}\right)}{\Gamma\left(\frac{2n+8}{2}\right)}\right]\Bigg/\left[\frac{\Gamma\left(\frac{2n+5}{2}\right)}{\Gamma\left(\frac{2n+6}{2}\right)}\right]$$

$$x=\frac{n+2}{n+3}\left[2\frac{(2n+7)(2n+5)}{(2n+8)(2n+6)}-\frac{2n+5}{2n+6}\right]a=\frac{(n+2)(2n+5)}{(n+3)(2n+8)}a.$$

Let $m=4$, then

$$x=\frac{n+2}{n+3}a\left[\frac{\Gamma\left(\frac{4n+13}{2}\right)}{\Gamma\left(\frac{4n+14}{2}\right)}-8\frac{\Gamma\left(\frac{4n+15}{2}\right)}{\Gamma\left(\frac{4n+16}{2}\right)}+8\frac{\Gamma\left(\frac{4n+17}{2}\right)}{\Gamma\left(\frac{4n+18}{2}\right)}\right]\Bigg/\left[\frac{\Gamma\left(\frac{4n+9}{2}\right)}{\Gamma\left(\frac{4n+10}{2}\right)}\right]$$

$$x=\frac{n+2}{n+3}\left[\frac{(4n+11)(4n+9)}{(4n+12)(4n+10)}-8\frac{(4n+13)(4n+11)(4n+9)}{(4n+14)(4n+12)(4n+10)}\right.$$

$$\left.+8\frac{(4n+15)(4n+13)(4n+11)(4n+9)}{(4n+16)(4n+14)(4n+12)(4n+10)}\right]a.\quad x=\frac{(n+2)(4n+9)(4n+11)}{(n+3)(4n+14)(4n+16)}a.$$

The first factor is always $\frac{n+2}{n+3}$, the numerator of the second factor is $mn+2m+1$, the denominator $mn+3m+2$, the numerator of each succeeding factor is 2 greater than the one immediately preceding, so also for each succeeding factor in the denominator.

The number of factors for $m=2$, is 2, for $m=4$, is 3, for $m=6$, is 4 and so on.

Let $m=6$, then $mn+2m+1=6n+13$, $mn+3m+2=6n+20$ and

$$\bar{x}=\frac{(n+2)(6n+13)(6n+15)(6n+17)}{(n+3)(6n+20)(6n+22)(6n+24)}a=\frac{(n+2)(2n+5)(6n+13)(6n+17)}{(n+3)(2n+8)(6n+20)(6n+22)}a.$$

Let $m=8$, then $mn+2m+1=8n+17$, $mn+3m+2=8n+26$ and

$$\bar{x}=\frac{(n+2)(8n+17)(8n+19)(8n+21)(8n+23)}{(n+3)(8n+26)(8n+28)(8n+30)(8n+32)}a,$$ and so on for any value of m.

The centroid is on the axis of symmetry. $\therefore\ \bar{y}=0$.

If m be a positive fraction the centroid will be the origin and $\bar{x}=0$, $\bar{y}=0$.
This follows from the fact that there are as many loops, (equal) arranged about the centre as the denominator of the fraction represented by m has integers in it. Hence, if $m=\frac{q}{p}$, then there are p equal loops around the centre.

A general rule is as follows:

When m is a positive odd integer

$$\bar{x}=\frac{n+2}{n+3}\left[\frac{(mn+2m+1)(mn+2m+3)(mn+2m+5)\ldots\ldots}{(mn+3m+1)(mn+3m+3)(mn+3m+5)\ldots\ldots}\right]a,$$ to $\frac{m+1}{2}$ factors inside the brackets.

When m is a positive even integer

$$\bar{x}=\frac{n+2}{n+3}\left[\frac{(mn+2m+1)(mn+2m+3)(mn+2m+5)\ldots\ldots}{(mn+3m+2)(mn+3m+4)(mn+3m+6)\ldots\ldots}\right]a,$$ to $\frac{m}{2}$ factors inside the brackets. In either case $\bar{y}=0$.

ARITHMETIC.

Conducted by B. F. FINKEL, Kidder, Missouri. All contributions to this department should be sent to him.

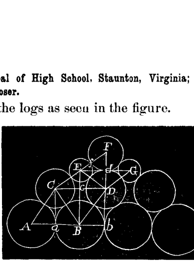
SOLUTIONS TO PROBLEMS.

9. Proposed by O. S. KIBLER, Superintendent of Schools, West Middleburgh, Logan County, Ohio.

Four logs of uniform thickness whose diameters are each four feet, lie side by side and touch each other. In the crevices of these logs lie three logs 3 feet in diameter, and in the crevices of the three logs lie two logs whose diameters are 2 feet. What must be the diameter of a log to lie on the top of the pile and touch the two logs and the middle one of the three logs?

Solution by Professor G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia; I L. BEVERAGE, Monterey, Virginia; and the Proposer.

Let A, B, C, D, E, F, G, be centres of the logs as seen in the figure.



Now $CD = AB = ab = EG$.

$\therefore\ Ed = \frac{1}{2}EG = 2$.

$EF^2 - 4 = dF^2,\ ED^2 - 4 = dD^2$,

$\sqrt{EF^2 - 4} + \sqrt{6\frac{1}{4} - 4} = DF$.

$EF = Ef + fF,\ DF = 1\frac{1}{2} + fF$,

$\sqrt{fF^2 + 2fF - 3} + \sqrt{2\frac{1}{4}} = 1\frac{1}{2} + fF$,

$\sqrt{fF^2 + 2fF - 3} = fF$,

$fF^2 + 2fF - 3 = fF^2$,

$fF = 1\frac{1}{2}$.

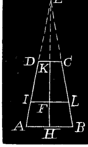
$\therefore$ the diameter required $= 3$ feet.

This problem was also solved by *A. L. FOOTE, JOHN T. FAIRCHILD, J. A. CALDERHEAD, H. C. WHITAKER, H. W. HOLYCROSS, P. S. BERG, CHARLES E. MYERS*, and *C. D. STILLSON*.

10. Proposed by MISS LEOTA MILLER, B. L., Professor of Natural Science and Art, Kidder Institute, Kidder, Missouri.

A carpenter is obliged to cut a board, that is in the form of a trapezoid, crosswise into two equivalent parts. The board is 12ft. long, 2 ft, wide at one end, and one foot wide at the other. How far from the narrow end must he cut?

Solution by B. F. FINKEL, Professor of Mathematics, Kidder Institute, Kidder, Missouri.



II.

1. Let $ABCD$ be the board.
2. $AB = 2$ feet $= b$, the width of the large end,
3. $DC = 1$ foot $= c$, the width of the small end, and
4. $HK = 12$ feet $= a$, the length of the board.
5. Produce HK, AD, and BC till they meet in E. Then by similar triangles,
6. $ABE : EIL : EDC :: AB^2 : IL^2 : DC^2$. But
7. $EIL = EDC + \frac{1}{2}(ABCD) = \frac{1}{2}(2\,EDC + ABCD) = \frac{1}{2}(EDC + EDC + ABCD) = \frac{1}{2}(EDC + EAB)$.
8. $\therefore\ IL^2 = \frac{1}{2}(AB^2 + DC^2) = \frac{1}{2}(b^2 + c^2)$.
9. $\therefore\ IL = \sqrt{\frac{1}{2}(b^2 + c^2)} = \sqrt{\frac{1}{2}(2^2 + 1^2)} = \frac{1}{2}\sqrt{10}$ ft., the dividing line.
10. Area of $ABCD = \frac{1}{2}(AB + CD) \times KH = \frac{1}{2}(b + c)a = 18$ sq. ft.
11. $\therefore$ Area of $ABIL = \frac{1}{2}ABCD = \frac{1}{4}(b + c)a = 9$ sq. ft.
12. But area of $DCIL = \frac{1}{2}(DC + IL) \times KF$ $= \frac{1}{2}[c + \sqrt{\frac{1}{2}(b^2 + c^2)}] \times KF = \frac{1}{2}(2 + \sqrt{10}) \times KF$.
13. $\therefore\ \frac{1}{2}(c + \sqrt{\frac{1}{2}(b^2 + c^2)}) \times KF = \frac{1}{4}(b + c)a$, whence
14. $KF = \dfrac{\frac{1}{2}(b + c)a}{[c + \sqrt{\frac{1}{2}(b^2 + c^2)}]} = \dfrac{18}{\frac{1}{2}(2 + \sqrt{10})} = \dfrac{36}{2 + \sqrt{10}} =$ $6.973666+$ feet.

III. $\therefore$ He must saw it in two at $6.973666+$ feet from the narrow end.

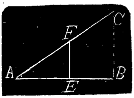
This problem was also solved by *G. B. M. Zerr, P. S. Berg, Charles E. Myers, J. A. Calderhead, A. L. Foote, H. C. Whitaker, H. W. Holycross, H. M. Cash* and *F. A. Swanger*.

11. Proposed by L. B. HAYWARD, Superintendent of Schools, Bingham. Ohio.

What length of rope will be required to draw water from a well, it being 38

feet to the water, the sweep to be supported by an upright post 20 feet high, and standing 20 feet from the well, and the foot of the sweep to strike the ground 20 feet from the foot of the upright post?

11. Solution by JOHN T. FAIRCHILD, Ada, Ohio.



AE=20 feet; FE, the post, =20 feet; BE =20 ft: and CB will be the required length of the rope.

By the similar triangles ABC and AEF, we have $AE:EF::AB:CB$, or 20 ft. : 20 ft. :: 40 feet : (CB =40 feet).

Also solved by *G. B. M. Zerr*, *A. L. Foote*, *H. C. Whitaker*, and *I. L. Beverage*.

12. Proposed by CHARLES E. MYERS, Canton, Ohio.

A man made his will to this effect: that if only the daughter returned home his wife should have $\frac{2}{3}$ and the daughter $\frac{1}{3}$ of the estate; and if only the son returned his wife should have $\frac{1}{3}$ and the son $\frac{2}{3}$. But the son and daughter both returned. How should the estate be divided?

I. Solution by M. A. GRUBER, A.M., War Department, Washington, D. C.

By the first condition, the wife's share =2 times the daughter's share.

By the second condition, the son's share =2 times the wife's share, hence 4 times the daughter's share.

The D's share = D's share;
The W's share =2 D's share;
· The S's share =4 D's share.

D's+W's+S's shares=7 D's share = the estate.

$\therefore$ D's share $=\frac{1}{7}$ of estate,
W's share $=\frac{2}{7}$ of estate,
S's share $=\frac{4}{7}$ of estate.

II. Solution by A. L. FOOTE, C. E., No. 80 Broad St., New York City.

Relatively the expectation of the son is double that of the mother, and the mother double that of the daughter, hence if we give the son four parts, the mother two parts and the daughter one part, we divide the estate into 7 equal parts and the daughter has $\frac{1}{7}$ of it, the wife $\frac{2}{7}$, and the son $\frac{4}{7}$.

There is, however, a legal aspect to this question. In the event of the son and daughter both returning the wife might legally claim her $\frac{1}{3}$ in which case the $\frac{2}{3}$ of the estate would be shared by the son and daughter in the ratio of 2 to 1 or the son would receive $\frac{4}{9}$ and the daughter $\frac{2}{9}$ of the estate.

Also solved by *G. B. M. Zerr*, *Robert J. Aley*, *H. C. Whitaker*, *W. F. Bradbury*, and *P. S. Berg*.

[NOTE.—This class of problems probably originated with the Romans whose laws of inheritance gave rise to numerous arithmetical problems. Professor Cajori, in his *History of Mathematics*, p. 80, quotes a problem involving the same principle and numbers as the one above. He further states that the celebrated Roman jurist, Salvianus Julianus, decided that the estate should be divided in the manner indicated by the above solutions.

According to modern jurisprudence, a will of this kind would very probably be set aside and an equal distribution of the estate be made.—ED.]

PROBLEMS.

19. Proposed by MISS LECTA MILLER, B. L., Professor of Natural Science and Art, Kidder Institute, Kidder, Missouri.

Bought sugar at $6\frac{1}{2}$ cents a pound, waste by transportation and retailing was 5%; interest on first cost to time of sale was 2%. How much must be asked per pound to gain 25%?

20. Proposed by L. B. HAYWARD, Superintendent of Schools, Bingham, Ohio.

I owed a merchant $600. The merchant agreed to take part of the amount and wait a year for the balance, if I would pay interest in advance. I paid $300. How much of this was interest on the unpaid balance, and how much went toward the payment of the debt?

21. Proposed by A. L. FOOTE, No 80, Broad St., New York City.

A merchant bought a certain quantity of corn for which he paid a certain sum of money; but on measuring he found only $\frac{38}{40}$ of the quantity he expected. He sold it gaining $\frac{1}{8}$ of the cost and received $2,160, which was at the rate of $12\frac{4}{13}$ cents per bushel more than he would have paid had he received the quantity expected. How many bushels did he suppose he had bought, and at what price?

[Selected from *Robinson's Arithmetical Problems.*]

[Solutions to the e problems should be received on or before June 1st.]

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS TO PROBLEMS.

9. Proposed by Professor G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

$$\left.\begin{array}{l} x+y^2+z^3=21 \\ x^2+y^3+z=45 \\ x^3+y+z^2=71 \end{array}\right\} \text{. Find } x,\ y, \text{ and } z.$$

Solution by the Proposer.

The equations may be written, as follows:

$$x-4+(y-3)(y+3)=(2-z)\ (4+2z+z^2)\ldots\ldots(1)$$

$$(x-4)(x+4)+(y-3)(9+3y+y^2)=2-z\ldots\ldots(2)$$

$$(x-4)(16+4x+x^2)+y-3=(2-z)(2+z)\ldots\ldots(3)$$

Let $16+4x+x^2=a$, $9+3y+y^2=b$, $4+2z+z^2=c$, $x+4=d$, $2+z=e$. Then from (1), (2), and (3), we get $x-4=c(2-z)-(y^2-9)\ldots\ldots(4)$,

$$x-4=\frac{2-z-b(y-3)}{d}\ldots\ldots(5),\quad x-4=\frac{e(2-z)-(y-3)}{a}\ldots\ldots(6).$$

Eliminating $(x-4)$ between (4) and (6), and (5) and (6), we get, $(ac-e)(2-z)=a(y^2-9)-(y-3)\ldots\ldots(7)$, $(a-de)(2-z)=(ab-d)(y-3)\ldots.(8)$. Eliminating $(2-z)$ between (7) and (8), we get, $a(y^2-9)(a-de)=a(y-3)(1+abc-dc-be)$, or $(y^2-9)\ A=(y-3)B$, suppose.

$$\therefore\ y^2-9=y\frac{B}{A}-3\frac{B}{A}\ ,\ \therefore\ y^2-\frac{B}{A}y=9-3\frac{B}{A}.$$

Completing the square root, $y^2-\frac{B}{A}y+\frac{B^2}{4A^2}=9-3\ \frac{B}{A}+\frac{B^2}{4A^2},\ \therefore\ y=3.$

Substituting value of y in (4) and (5),

$$x-4=c(2-z)\ldots.(9),\ d(x-4)=2-z\ldots.(10).$$

Eliminating $(2-z)$ between (9) and (10), $\frac{x-4}{c}=d(x-4)=x^2-16,$

$$\therefore\ x^2-\frac{x}{c}=16-\frac{4}{c}.\quad \therefore\ x^2-\frac{x}{c}+\frac{1}{4c^2}=16-\frac{4}{c}+\frac{1}{4c^2},\ \therefore\ x=4.$$

Values of x and y in $x^2+y^3+z=45$, gives $z=2$.

$\therefore\ x=4,\ y=3,\ z=2.$

Also solved by *Professor W. F. Bradbury.*

10. Proposed by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pennsylvania.

$$x^2+y^2+w^2+z^2=65\ldots.(1),$$
$$(x+z)^2+(y+w)^2=113\ldots.(2),$$
$$(y+z)^2+(x+w)^2=117\ldots.(3),$$
$$(x+y)^2+(z+w)^2=125\ldots.(4).$$

How many values has each of the four unknown quantities?

Solution by W. F. BRADBURY, A. M., Head-Master Cambridge Latin School, Cambridge, Massachusetts.

From (1) subtract (2), (3), and (4), successively,

$$2xz+2yw=48\ldots.(5),$$
$$2yz+2xw=52\ldots.(6),$$
$$2xy+2zw=60\ldots.(7).$$

Adding (5), (6), (7), and (1), we get,

$$(x+y+z+w)^2=225\ldots.(8),\quad x+y+z+w=\pm15\ldots.(9).$$

Using only + values, $x+z=15-(y+w)\ldots.(10)$.

Substituting in (2), $225-30(y+w)+(y+w)^2+(y+w)^2=113\ldots.(11)$, $(y+w)^2-15(y+w)=-56\ldots.(12)$,

$$y+w=\frac{15}{2}\pm\sqrt{\frac{225}{4}-\frac{224}{4}}=\frac{15}{2}\pm\frac{1}{2}=8,\text{ or }7\ldots.(13).$$

Hence from (10), $x+z=7$, or 8. In like manner, substituting from (9) in (3) and (4), we find $y+z=6$, or 9; $x+w=9$, or 6; $x+y=5$; $z+w=10$.

From these we find, $x=3$, or 2, $y=2$, or 3. $w=6$, or 4, $z=4$, or 6. Using the negative values other answers can be found.

[There are in all 16 values for each of the unknown quantities, arising from the reduced equations $x+y+z+w=\pm15$, $x+y-z-w=\pm5$, $x+y+z-w=\pm1$, $x-y-z+w=\pm3$, as follows:

$$x=\pm6,\ \pm4,\ \pm2,\ \pm3,\ \pm4\tfrac{1}{2},\ \pm5\tfrac{1}{2},\ \pm3\tfrac{1}{2},\ \pm1\tfrac{1}{2}.$$
$$y=\pm4;\ \pm6,\ \pm3,\ \pm2,\ \pm5\tfrac{1}{2},\ \pm4\tfrac{1}{2},\ \pm1\tfrac{1}{2},\ \pm3\tfrac{1}{2}.$$
$$z=\pm2,\ \pm3,\ \pm6,\ \pm4,\ \pm3\tfrac{1}{2},\ \pm1\tfrac{1}{2},\ \pm4\tfrac{1}{2},\ \pm5\tfrac{1}{2}.$$
$$w=\pm3,\ \pm2,\ \pm4,\ \pm6,\ \pm1\tfrac{1}{2},\ \pm3\tfrac{1}{2},\ \pm5\tfrac{1}{2},\ \pm4\tfrac{1}{2}.$$—EDITOR.

Also solved by Professor *G. B. M. ZERR.*

PROBLEMS.

19. Proposed by A. L. FOOTE, C. E., No. 80, Broad Street, New York City.

$$\text{Given } \left.\begin{array}{l} \dfrac{xyz}{\sqrt[n]{(x^n+y^n)}}=a \\ \dfrac{xyz}{\sqrt[n]{(x^n+z^n)}}=b \\ \dfrac{xyz}{\sqrt[n]{(y^n+z^n)}}=c \end{array}\right\}. \text{ To find } x,\ y, \text{ and } z.$$

20. Proposed by J. K. ELLWOOD, A. M., Principal of Colfax School, Pittsburg, Pennsylvania.

At what price must the government sell 5% \$100 bonds to run 10 years, interest payable annually, to make them equivalent to 3% BONDS AT PAR to run 10 years, interest payable annually? At what price if interest be paid semi-annually?

GEOMETRY.

Conducted by B. F. FINKEL, Kidder, Missouri. All contributions to this department should be sent to him.

SOLUTIONS TO PROBLEMS.

6. Proposed by EARL D. WEST, West Middleburg, Logan County, Ohio.

Having given the sides 6, 4, 5, 3 respectively of a trapezium, inscribed in a circle, to find the diameter of the circle.

Solution by Hon. JOSIAH H. DRUMMOND, LL.D., Portland, Maine.

The following solution I discovered (so far as I know) and wrote on the margin of one of the pages in Young's Analytical Geometry containing his solution, January 12, 1873.

Conceive the trapezium, $ABDC$, inscribed in the circle and draw the diagonals. Let $CD=6$, $BD=4$, $AB=5$ and $AC=3$, and AD and BC be the diagonals.

Since the product of the diagonals=the sum of the products of the opposite side (Young's Geom. p. 211) we have $AD\times BC=30+12=42$....(1). Also since $CD\times BD+AB\times AC$: $CD\times AC+AB\times BD$ :: AD : BC. (Young's Geom. p. 212) we have 39 : 38 :: $AD:BC$....(2). Substituting in (1) the value of AD as found in (2) we readily find $BC=2\sqrt{\frac{133}{13}}$. In the triangle BCD we have the three sides and readily find area $=\frac{48}{13}\sqrt{10}$. But diameter of circumscribing circle=one-half of the continuous product of three sides divided

by area; hence $D=\frac{1}{2}\times6\times4\times2\sqrt{\frac{133}{13}\div\frac{48}{13}}\sqrt{10}=\sqrt{43.225}=6.57457223-$.

7. **Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy in the Ohio University, Athens. Ohio.**

Through each point of the straight line $x=my+h$ is drawn a chord of the parabola $y^2=4ax$, which is bisected in the point. Prove that this chord touches the parabola $(y+2am)^2=8a(x-h)$.

Solution by the Proposer.

Let the chord be $gx+fy=1\ldots.(1)$.

This cuts the curve $y^2=4ax\ldots.(2)$, in the points whose co-ordinates are given by the equations

$$x^2-\frac{2(g+2af^2)}{g^2}x+\frac{1}{g^2}=0\ldots.(3), \text{ and}$$

$$y^2+\frac{4af}{g}y-\frac{4a}{g}=0\ldots.(4).$$

The middle of the chord is then $\left(\frac{g+2af^2}{g^2},-\frac{2ag}{f}\right)$.

If this point be on the line $x=my+h\ldots.(5)$,

$$\frac{g+2af^2}{g^2}=-\frac{2amf}{g}+h\ \ldots(6),$$

$$\text{or, } g+2af^2=-2amfg+hg^2\ldots.(7).$$

Making this homogeneous by aid of (1),

$$(h-x)\frac{g^2}{f^2}-(y+2am)\frac{g}{f}-2a=0\ldots.(8), \text{ a quad-}$$

ratic in the undetermined constant $\frac{g}{f}$, and giving the envelope $(y+2am)^2=8a(x-h)$.

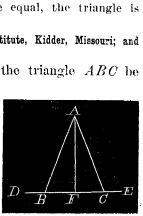
Also solved by L. E. Pratt, Alfred Hume, G. B. M. Zerr, and J. F. W. Scheffer.

8. **Proposed by ADOLPH BAILOFF, Durand, Wisconsin,**

If the two exterior angles at the base of a triangle are equal, the triangle is isosceles.

Solution by MISS GRACE H. GRIDLEY, Student in Kidder Institute, Kidder, Missouri; and P. S. BERG, Apple Creek, Ohio.

Let the exterior angles, ABD and ACE of the triangle ABC be equal. To prove the triangle isosceles.



PROOF: $\angle ACE+\angle ACF=2$ rt. $\angle s$.

Also $\angle ABD+\angle ABF=2$rt. $\angle s$.

$\therefore\ \angle ACE+\angle ACF=\angle ABD+\angle ABF$.

(Things equal to the same thing are equal to each other).

But $\angle ACE=\angle ABD$. (HYP.)

$\therefore \angle ACF=\angle ABF$.

(If equals be subtracted from equals, the remainders are equal).

$\therefore$ Side AB=side AC, being sides opposite equal angles, and therefore $\triangle\ ABC$ is isosceles. Q. E. D.

Also solved by *J. A. Calderhead, J. R. Baldwin, Josiah H. Drummond, H. M. Cash, J. F. W. Scheffer*, and *G. B. M. Zerr.*

PROBLEMS.

28. Proposed by Professor HENRY HEATON, M. S., Atlantic, Iowa.

Through three given points to pass two spherical surfaces tangent to a given sphere.

29. Proposed by H. W. HOLYCROSS, Superintendent of Schools, Pottersburg, Ohio.

If the two angles at the base of a triangle are bisected; and through the point of meeting of the bisectors a line is drawn parallel to the base, the length of the parallel between the sides is equal to the sum of the segments of the sides between the parallel and the base.

30. Proposed by CHARLES E. MYERS, Canton, Ohio.

A circle containing one acre is cut by another whose center is on the circumference of the given circle, and the area common to both is one-half acre. Find the radius of the cutting circle.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS TO PROBLEMS.

7. Proposed by Professor J. F. W. SCHEFFER, A. M., Hagerstown, Maryland.

To determine the function $F(x)$ so that $F(x+y)\times F(x-y)=[F(x)]^2-[F(y)]^2$.

Solution by the Proposer.

Putting $x+y=z$, $x-y=t$, we get

$F(z).\ F(t)=[F(x)]^2-[F(y)]^2$. Differentiating this equation twice according to z and t as independent variables, and considering that $\frac{dx}{dz}=\frac{1}{2}, \frac{dx}{dt}=\frac{1}{2}, \frac{dy}{dz}=\frac{1}{2}, \frac{dy}{dt}=-\frac{1}{2}$; we obtain $F''(z)F(t)F''(t).F(z)$.

$\therefore \frac{F''(z)}{F(z)}=$ constant $=a^2$. Denoting $F(z)$ or $F(x)$ by u, we have the differential equation $\frac{d^2u}{dx^2}=a^2u$. $\therefore\ u=Ce^{ax}+C'e^{-ax}$. Since $F(0)=0$, we have $C'=-C$, $\therefore u=C(e^{ax}-e^{-ax})=C\sin bx$, where C and b designate any two constant quantities.

8. A woodman fells a tree 2 feet in diameter, cutting half way through from each side. The lower face of each cut is horizontal, and the upper face makes an angle of 45° with the horizontal. How much wood does he cut out?

[Selected from *Byerly's Integral Calculus.*]

I. Solution by CHARLES E. MYERS, Canton, Ohio.

Conceive each part removed to be generated by the motion of a right-angled triangle, moving so that its base is in a cross section of the tree and perpendicular to the axis of the tree, the sides of the triangle varying and the angles remaining constant=45°. Let z=the altitude of the triangle at any time; V=$2v$=entire volume removed, and put $2r=2$ feet.

Putting the origin of co-ordinates at the circumference, we have, $v=\int_0^{2r}\frac{1}{2}yzdx$. From the circle, $y\ (2r-x)^{\frac{1}{2}}$, and since the angle=45°, $z=y$ at all times. Substituting these values of y and z in the above equation and integrating, we have,

$v=\int_0^{2r}\frac{1}{2}(2rx-x^2)dx=\frac{2}{3}r^3$, and doubled, gives $V=\frac{4}{3}r^3=1\frac{1}{3}$ cu. ft.

II. Solution by J. R. BALDWIN, A. M., Professor of Mathematics in the Davenport Business College, Davenport, Iowa.

Let $r=1$ foot=the radius of the tree, AB the common edge of the two cuts, θ=the angle which the radius makes with AB, $\varphi=45°$. For the element of volume, we have, $2r\cos\theta.r\sin\theta\tan\varphi.d(r\sin\theta)$.

$$\therefore \text{ Volume } =4r^3\tan\varphi\int_0^{\frac{1}{2}\pi}\cos^2\theta\sin\theta d\theta=-4r^3\left[\frac{\cos^3\theta}{3}\right]_0^{\frac{1}{2}\pi}=\frac{4}{3}r^3=1\tfrac{1}{3}\text{cu.ft.}$$

Also solved by *M. C. Stevens, P. H. Philbrick, Seth Pratt, Alfred Hume, H. W. Draughon, H. C. Whitaker, G. B. M. Zerr, W. L. Harvey*, and *P. S. Berg*.

PROBLEMS.

16. Proposed by F. P. MATZ, M. Sc.,Ph.D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Differentiate $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ with regard to $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

17. Proposed by H. W. DRAUGHON, Clinton, Louisiana.

To find the volume generated by revolving a circular segment whose base is a given chord, about any diameter as an axis.

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* *

MECHANICS.

Conducted by B. F. FINKEL. Kidder, Missouri. All contributions to this department should be sent to him.

SOLUTIONS TO PROBLEMS.

3. Proposed by CHARLES E. MYERS, Canton, Ohio.

A spherical air-bubble, having risen from a depth of 1,500 feet in water, was one inch in diameter when it reached the surface; what was its diameter at the point of starting?

I. Solution by the Proposer.

Let x=radius at the point of starting. At the surface the pressure per unit is equal to a column of water 34 feet high, and at a depth of 1500 feet by 1500+34, and since the volumes are inversely as the pressures, we have,

$\frac{4}{3}\pi\left(\frac{1}{2}\right)^3 : \frac{4}{3}\pi x^3 :: 1500+34:34$, or, $6136x^3=17$, whence, $x=.14$ inches and $2x=.28$ inches, the required diameter.

II. Solution by Professor G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

Let p=pressure per square inch on the bubble as it emerges at the surface, and p' the pressure per inch on the bubble at the bottom of the 1500 feet of water;then if the temperature does not change and assuming that Boyle's law holds good, namely, that the volume of a given mass of gas, at a constant temperature, is inversely as the pressure, the required diameter$=(p\div p')$.

But both p and p' are influenced by temperature, by atmospheric pressure, which is always changing, by the distance of the place of observation from the earth's centre, which affects the weight of both air and water.

Also the compressibility of the water affects the amount in a column of 1500 feet; and if sea-water is considered, another important factor enters into the calculation as sea-water is heavier and less compressible than pure water.

Let us assume the truth of Boyle's law and also of Charles's law, that the product of the volume and pressure of any gas is proportional to the absolute temperature,calculated from -460°F. Assuming also that the temperature of the water is constant throughout the course of the bubble, and, that the weight of a body above the surface of the earth is inversely as the square of its distance from the centre, but below the surface directly as the distance; that the compressibility of pure water is .000048 per atmosphere of 14.7 lbs. to the square inch; that the compressibility of sea-water is $.925\times.000048=.000044$ per atmosphere and its specific gravity is 1.0263; we may proceed as follows:

Let r=the distance of the centre of the earth from the surface of the water; t=temperature of the bubble at bottom, t'=the temperature of it at the top; h=height in feet of a column of water of one-inch cross-section and weighing 14.7 lbs; h'=height of a column of water rising from the neighborhood of the emerged bubble,and supported exactly by the atmospheric pressure at the particular time and place. Then, approximately, the standard atmospheric pressure is to the pressure p as

$$\int_o^h \frac{r^2}{(r+h-x)^2} \cdot \frac{h^2 dx}{h^2 - .000048 \int_o^x \frac{r^2 dy}{(r+h-y)^2}} : \int_o^{h'} \frac{r^2}{(r+h'-x)^2} \cdot$$

$$\frac{h^2 dx}{h^2 - .000048 \int_o^x \frac{r^2 dy}{(r+h'-y)^2}}$$

for pure water and the same formula for sea water by putting .000044 for .000048 with different values for h and h'. However we will use the ordinary formula $h:h'=14.7:p$, or $p=\frac{14.7h'}{h}$.

We next have $p:p'=h':k$, where

$$k=h'+\int_0^{1500} \frac{r-x}{r} \cdot \frac{h^2 dx}{h^2 - .000048\left(h'+\int_0^x \frac{r-y}{r} dy\right)}$$

$$= h' + \int_0^{1500} \frac{2h^2(r-x)dx}{2r(h^2-000048h')-.000096rx+.000048x^2},$$

$$= \frac{h'}{.000048}\Big[\log 2r(h^2-.000048h') - \log\left\{ 2r(h^2-.000048h')-.144r+108 \right\}\Big],$$

for pure water,

$$\frac{h'}{.000044}\Big[\log 2r(h_1^2-.000044h_1')-\log\left\{ 2r(h_1^2-.000044h_1')-.132r+99 \right\}\Big]$$

for sea water.

The semi-axes of the earth are $a=20926202$ feet, $c=20854895$ feet and we obtain r for any latitude θ from the formula $r=\frac{ac}{(a^2\sin^2\theta+c^2\cos^2\theta)^{\frac{1}{2}}}$ for Greenwich $\theta=51^\circ 21' 38\frac{1}{2}''$ and $r=20882610$. If the surface of the water in question be any where in this latitude and at sea-level, let $t'=(460+60)^\circ$F, $t=(460+75)^\circ$F; also let $h=33.35$, $h'=32.90$ for fresh water; then for sea-water $h_1=32.50$ $h_1'=32.057$; hence $p=14.502$ lbs. and $k=32.90+23171302$ (log 46452159574 $-$ log 46449152586) $=1532.9502$ for pure water, $k'=32.06+24005682$ (log 44107504286 $-$ log 44104747880) $=1531.6325$ for sea water.

Now from $p:p'=h':k$, $k'=675.679$ lbs. for pure water.

$p/p'=.0214629$. $p'=692.882$ lbs. for sea water.

$p/p'=.0209297$. $\frac{t}{t'}=\frac{535}{520}=1.028846$

$\therefore$ for fresh water, diameter $=\{(.0214629)(1.028846)\}^{\frac{1}{3}}=.28055161$;
for sea-water diameter $=\{(.02092997)(1.028846)\}^{\frac{1}{3}}=.27817051$.

If the temperature does not change then
for pure water the diameter $=(.0214629)^{\frac{1}{3}}=.27790474$,
for sea-water the diameter $=(.02092997)^{\frac{1}{3}}=.27554582$.

DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS TO PROBLEMS.

3. Proposed by O. S. KIBLER, Superintendent of Schools, West Middleburg, Logan County Ohio.

It is required to find three whole numbers in an arithmetical progression such that the sum of every two of them shall be a square.

II. Solution by ARTEMAS MARTIN, A. M., Ph.D., LL.D., U. S. Coast and Geodetic Survey Office, Washington, D. C.

Let $x-y$, x and $x+y$ denote the required numbers in arithmetical progression. Then must

$2x-y=\square=a^2\ldots.(1)$, $2x=\square=b^2\ldots.(2)$, $2x+y=\square=c^2\ldots.(3)$.

Substituting $x=\frac{1}{2}b^2$, the value given by (2), in (1) and (3) we get

$$b^2-y=\square=a^2\ \ldots(4),\ \text{and}\ b^2+y=\square=c^2\ldots.(5).$$

From (4) and (5) we find

$$y=b^2-a^2=c^2-b^2\ldots.(6),\ \text{therefore}\ 2b^2=a^2+c^2\ldots.(7),$$

which is the only condition remaining to be satisfied.

Let $c=m+n$, and $a=m-n$, then (7) becomes

$$b^2=m^2+n^2\ldots.(8),$$

which is satisfied by assuming $m=2pq$, $n=p^2-q^2$,

Hence $x=\frac{1}{2}b^2=\frac{1}{2}(m^2+n^2)=\frac{1}{2}(p^2+q^2)^2$,

$$y=b^2-a^2=(m^2+n^2)-(m-n)^2=2mn=4pq(p^2-q^2),$$

and the required numbers are

$$x-y=\tfrac{1}{2}(m^2+n^2)-2mn=\tfrac{1}{2}(p^2+q^2)^2,-4pq(p^2-q^2),$$

$$x=\tfrac{1}{2}(m^2+n^2)=\tfrac{1}{2}(p^2+q^2)^2,$$

$$x+y=\tfrac{1}{2}(m^2+n^2)+2mn=\tfrac{1}{2}(p^2+q^2)^2+4pq(p^2-q^2).$$

Taking $p=5$, $q=4$, we get $x=840\frac{1}{2}$, $y=720$; hence $x-y=120\frac{1}{2}$, $x+y=1560\frac{1}{2}$, and multiplying by 4 for integers the required numbers are found to be 482, 3362 and 6242.

This set of numbers is the same as that found by different methods of solution in Maynard's edition of the key to Bonnycastle's Introduction to Algebra, published in London in 1835. See pp. 113-115.

An infinite number of other sets may be found.

4. Proposed by H. W. HOLYCROSS, Superintendent of Schools, Pottersburg, Union Co., Ohio.

What value of x will render $4x^4+12x^3-3x^2-2x+1$ a square?

II. Solution by ARTEMAS MARTIN, A.M., Ph.D., LL.D., U. S. Coast and Geodetic Survey Office, Washington, D. C.

Put $4x^4+12x^3-3x^2-2x+1=(2x^2+3x-3)^2$,

$$=4x^4+12x^3-3x^2-18x+9;$$

whence $x=\frac{1}{2}$. *Other values may be found.*

[Dr. Martin also sent excellent solutions to Nos. 1 and 2. R. H. Young, of West Sunbury, Pa., and Alvin E. Schmidt, Winesburg, Ohio, sent solutions to 1, 3

and 4. These solutions were not received in time to be acknowledged in March No.—ED.]

PROBLEMS.

9. Proposed by ISAAC L. BEVERAGE, Monterey, Virginia.

It is required to find three numbers the sum of whose 4th power is a square.

10. Proposed by L. B. HAYWARD, Bingham, Ohio.

Find two numbers such that each of them and also their sum and their difference when increased by unity shall all be square numbers.

AVERAGE AND PROBABILITY

Conducted by B. F. FINKEL, Kidder, Mo. All contributions to this department should be sent to him.

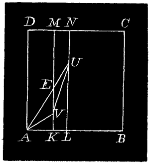
SOLUTIONS TO PROBLEMS.

2. Proposed by O. S. KIBLER, Superintendent of Schools, West Middleburg, Logan Co., Ohio.

Find the average area of a triangle formed by joining an angle of a square with any two points within the square.

Solution by Professor G. B. M. ZERR, Principal of High School, Staunton, Virginia.

Let $ABCD$ be the square side a, and U, V, the two random points.



Through V, U draw KM, NL, parallel to AD, KM meeting AU in E.

Let $AL=X$, $AK=w$, $LU=y$, $KV=z$, $KE=z'$.

Then $z'=\dfrac{wy}{x}$; also

area $AUV=\frac{1}{2}(wy-xz)=u$, when $z<z'$,

area $AUV=\frac{1}{2}(xz-wy)=u_1$, when $z>z'$.

The limits of x are 0 and a; of w, 0 and x; of y, 0 and a; of z, 0 and z', and z' and a.

Hence, the required average area is

$$\varDelta=\frac{\displaystyle\int_0^a\int_0^x\int_0^a\left\{\int_0^{z'} udz+\int_{z'}^a u_1dz\right\}dxdwdy}{\displaystyle\int_0^a\int_0^x\int_0^a\int_0^a dxdwdydz.}$$

$$=\frac{2}{a^4}\int_0^a\int_0^x\int_0^a\left\{\int_0^{z'} udz+\int_{z'}^a u_1dz\right\}dxdwdy.$$

$$=\frac{1}{2a^4}\int_0^a\int_0^x\int_0^a\left(\frac{2w^2y^2}{x}+a^2x-2awy\right)dxdwdy,$$

$$=\frac{1}{6a}\int_o^a\int_o^x\left(\frac{2w^2}{x}+3x-3w\right)dx\,dw=\frac{1}{36a}\int_o^a 13x^2\,dx=\frac{13a^2}{108}.$$

This problem was also solved by Professors *Matz* and *Droughon*.

BOOKS.

An Examination Manual in Plane Geometry. By G. A. Wentworth and G. A. Hill, 8 vo., cloth, 138 pp. Price, $0.50. Boston: Ginn & Co.

This little manual is designed to give elementary instruction in the art of handling origional theorems and problems to supply a series of graded test-papers which can be used not merely as tests of knowledge actually obtained, but also as means of developing and strengthening the power to originate and carry on a logical train of thought. Teachers of geometry will find this little volume quite interesting and useful.

A History of Mathematics. By Florian Cajori, Formerly Professor of Applied Mathematics in the Tulane University of Louisiana, now Professor of Physics in Colorado College. 8 vo., cloth, 422 pp. Price $3.50. New York: Macmillan & Co.

After having read this admirable work, I take great pleasure in recommending it to all students and teachers of mathematics. The development and progress of mathematics have been traced by a master pen and in the true spirit of the historian from its humble origin on the banks of the Nile down to the present time.

Who does not want to know something of that science to which more than to any other single branch of human knowledge, we owe our present state of civilization? Every mathematician should procure a copy of this work. Students who have a taste for mathematics should study this book and be inspired to higher aims and nobler asperations by the efforts and achievements of Descartes, Pascal, Fermat, Newton, Euler, Lagrange, Laplace, and hundreds of others who have enriched the science of mathematics a thousand fold by their genius.

Professor Cajori has carefully garnered the grain and burned the chaff. The book is written in a clear and pleasing style.

Colleges and Universities should not be slow to include this work as a text-book to be persued in the mathematical course, not to add a greater burden to the already over taxed brain of the student but to relieve it from the monotony of routine class work.

In some future issue of the MONTHLY, Dr. Halsted will give an article on this important book.

ERRATA.

p. 111, (in some copies of this issue) 3d line in first paragraph handsomly should read handsomely.

p. 113, (in some copies of this issue) last line in 1st paragraph *cousa* should read *causa.*

p. 113, (in some copies of this issue) 3d line in 5th paragraph *sequoitur* should read *sequitur.*

p. 123, near middle of the page, $\therefore\ \bar{x}=0$ should read $\bar{y}=0$.

p. 123, near middle of page, interger should read *integer.*